

# THE NEWTONIAN CORRESPONDENCE

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It is shown that the field equations of general theory of relativity in the Einstein tensor form and the unimodular theory of gravity do not fulfill the correspondence principle commitment completely. The consistent formalisms are briefly discussed.

Key words: correspondence principle, general relativity, unimodular gravity.

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## 1. INTRODUCTION

General relativity (GR) is established upon some physical principles. The principle of covariance (PC) and the correspondence principle (CP) are two of them which are considered in this letter. According to PC the field equations should have tensorial form, and GR must agree with the Newtonian gravitational theory in the limit of weak gravitational fields and low velocities by CP. Different aspects of the Newtonian limit may be classified as follows:

- a - The equation of geodesic deviation.
- b - The geodesic equation.
- c - The weak field limit of GR should give the same equations of motions as Newtonian gravity.

In GR we are dealing with second rank tensorial field equations, generally a set of ten relations, while in the Newtonian gravity we have only one Poisson equation and it seems there is no correspondence for nine of the rest. Does this mean that the PC breaks in taking the Newtonian limit? The answer is negative. In the Newtonian limit the Lorentz transformations reduce to Galileo transformations, so that  $t$  appears as a scalar. By Newtonian correspondence we must consider weak fields and low velocities. It turns out that in the spatial components of the field equations the first non-zero term has an order of approximation higher than the corresponding one in the  $tt$ -component. Since in finding the Newtonian limit we merely keep the first order terms in the  $tt$ -component, this leads to  $0 = 0$  for other components. For more clarification we may work in a system of units that  $c \neq 1$ . This explicitly shows, when the velocity of light tends to infinity, how some components of the field equation disappear. From this point of view we may say that PC is not violated but the other components have no physical information. So we may restate the item (c) as follows:

ć - The (00)-component of the field equation must reduce to the Poisson equation for a weak stationary field produced by nonrelativistic matter[1].

We are going to show that, from (ć) point of view, the GR field equations in the form of Einstein tensor and the field equations of the unimodular gravity do not satisfy CP. The consistent form and its consequences are discussed.

## 2. EINSTEIN TENSOR FORM

We restrict our discussion to the Schwarzschild space which is the solution of the field equations for spherically symmetric vacuum space around a point mass M. In the literature we have the Einstein field equations in the form Einstein tensor proportional to energy-momentum tensor i.e. :

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} \quad (1)$$

satisfying CP so that the (00)-component of the field equation reduces to the Poisson equation in the weak field limit. It will be shown that for the Schwarzschild metric this is not so.

The Weinberg's argument to reach this result is based on the fact that in a nonrelativistic system  $T_{ij} \ll T_{00}$ , then  $|G_{ij}| \ll |G_{00}|$  and  $R_{ij} \approx \frac{1}{2}g_{ij}R$ . Furthermore  $g_{\alpha\beta} \approx \eta_{\alpha\beta}$  and the curvature scalar is given by

$$R \approx R_{kk} - R_{00} \approx 2R_{00} \quad (2)$$

So concludes that  $G_{00} \propto R_{00}$  and  $R_{00} \propto \nabla^2 g_{00}$  [2]. The weak point in this argument is that by making use of  $G_{ii} = 0$  in calculating  $G_{tt}$  actually different components of the field equations are combined. In other words the Poisson equation is constructed by a proper mixing of all the available equations. This is in contrast with the original claim that the  $(tt)$ -component of the field equation in the weak field limit gives the Poisson equation.

The Einstein tensor in the weak field limit may yield to  $\square\psi_{\mu\nu}$  , where  $\psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$  ,  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$  in Minkowski coordinate , provided that the Einstein condition is satisfied as follows [3]

$$h_{\nu,\mu}^\mu - \frac{1}{2}h_{,\nu} = 0 \quad , \quad h = \eta^{\mu\nu}h_{\mu\nu} \quad (3)$$

If this condition holds , in the stationary case  $\square$  reduces to  $\nabla^2$  and the Poisson equation is obtained automatically. Let us see what happens in the weak field limit of Schwarzschild metric. We have

$$h_{tt} = \frac{2\phi}{c^2} \quad , \quad h_{x_i x_i} = \frac{2\phi x_i^2}{r^2 c^2} \quad , \quad h_{x_i x_j} = \frac{2\phi x_i x_j}{r^2 c^2} \quad , \quad i, j = 1, 2, 3$$

$$-\phi = \frac{GM}{r} \ll c^2. \quad (4)$$

Using (4) we get  $h = 0$  and these do not satisfy (3), i.e. Einstein condition does not hold in this case. It means that for Schwarzschild space we do not end to the Poisson equation in the weak field limit.

Since curvature tensor and its contractions are invariant quantities under a gauge transformation of  $h_{\mu\nu}$  as follows

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon\xi^\mu$$

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - 2\xi_{(\mu,\nu)} \quad (5)$$

it is possible to find a gauge in which Einstein condition holds. This gauge may be obtained from

$$\square\xi_\nu = \psi^\mu_{\nu,\mu} \quad , \quad \psi^\mu_{\nu,\mu} = h^\mu_{\nu,\mu} - \frac{1}{2}h_{,\nu} \quad (6)$$

We may conclude that the weak field limit of GR and Newtonian field equation are not in the same gauge.

In what follows we will see that this approximation although may lead to a correct prediction of reciprocal of distance for Newtonian potential but indeed does not reduce to the Poisson equation as is required.

The line element of a spherically symmetric vacuum space is

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (7)$$

where  $B(r) = A^{-1}(r) = 1 + \frac{2\phi}{c^2}$ . Using (7) the nonvanishing components of Einstein tensor are:

$$G_{rr} = -\frac{B'}{rB} + \frac{A-1}{r^2} \quad (8)$$

$$G_{\theta\theta} = -\frac{r^2B''}{2AB} + \frac{r^2B'}{AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) \quad (9)$$

$$G_{\varphi\varphi} = \sin^2\theta G_{\theta\theta} \quad (10)$$

$$G_{tt} = c^2\left[-\frac{BA'}{rA^2} + \frac{B}{r^2}\left(-1 + \frac{1}{A}\right)\right] \quad (11)$$

prime stands for differentiation with respect to  $r$ .

In contrast to what is expected,  $G_{tt}$  for the Schwarzschild metric merely contains the first order differentiation with respect to  $r$  and in no way can yield to the Poisson equation in weak field limit. Therefore there is an obvious discrepancy between the obtained result and the Newtonian equation. Although (11) in the limit of weak fields gives

$$G_{tt} \cong 2\left(\frac{\phi'}{r} + \frac{\phi}{r^2}\right) \quad (12)$$

which has the same solution of reciprocal of  $r$  as the Poisson equation possess for a particle with mass  $M$ . This can be considered as a gauge violation of CP which may be forbidden too.

### 3. UNIMODULAR GRAVITY

In a more plausible consideration of cosmological constant as an integration constant the unimodular gravity is actually very well motivated.

If the determinant of  $g$  is not dynamical then the action only has to be stationary with respect to variations in the metric for which  $g^{\mu\nu}\delta g_{\mu\nu} = 0$ , yielding the field equations [4,5,6]

$$R^{\mu\nu} - \frac{1}{4}g^{\mu\nu}R = -\frac{8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{4}g^{\mu\nu}T^\alpha_\alpha) \quad (13)$$

with  $T^{\mu\nu}$  as conserved stress tensor of matter. The combination of this with Bianchi identities for the covariant derivative of the Einstein tensor gives a nontrivial consisting condition

$$\frac{1}{4}\partial_\mu R = \frac{8\pi G}{c^4}(\frac{1}{4}\partial_\mu T^\lambda_\lambda) \quad (14)$$

Denoting the constant of integration by  $-4\Lambda$  the Einstein field equations is recovered.

We also see that this form of field equations i.e. (13), regrettfully does not satisfy the CP from (ć) point of view. For spherically symmetric vacuum space (7) the components of (13) are :

$$R_{rr} - \frac{1}{4}g_{rr}R = \frac{B''}{4B} - \frac{B'}{8B}(\frac{A'}{A} + \frac{B'}{B}) + \frac{A-1}{2r^2} - \frac{A'}{rA} \quad (15)$$

$$R_{\theta\theta} - \frac{1}{4}g_{\theta\theta}R = -\frac{r^2B''}{4AB} + \frac{r^2B'}{8AB}(\frac{A'}{A} + \frac{B'}{B}) + \frac{1}{2}(\frac{1}{A} - 1) \quad (16)$$

$$R_{\varphi\varphi} - \frac{1}{4}g_{\varphi\varphi}R = \sin^2\theta (R_{\theta\theta} - \frac{1}{4}g_{\theta\theta}R) \quad (17)$$

$$R_{tt} - \frac{1}{4}g_{tt}R = c^2[-\frac{B''}{4A} + \frac{B'}{8A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B}{2rA}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B}{2r^2}(1 - \frac{1}{A})] \quad (18)$$

In the weak field limit for the (18) we get

$$R_{tt} - \frac{1}{4}g_{tt}R = -\frac{\phi''}{2} + \frac{\phi}{r^2} \quad (19)$$

Again for (19) we have reciprocal of  $r$  as its solution but it is not the Poisson equation as is expected from CP.

#### 4. CONSISTENT FORM

The ordinary field equations in the following form

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \quad (20)$$

fulfill the CP requirement, that is the (00)- component of (20) for weak field limit of Schwarzschild metric reduces to

$$-\phi'' - \frac{2\phi'}{r} = -4\pi GM\delta(\vec{r}) \quad (21)$$

which in a compact form is exactly the Poisson equation

$$\nabla^2\phi = 4\pi GM\delta(\vec{r}) \quad (22)$$

For a perfect fluid the (00)-component of the RHS of (20) in the weak field limit reduces to

$$4\pi G(\rho + 3p/c^2) \quad (23)$$

which is equal to  $8\pi G\rho_t$  where  $\rho_t$  is the timelike convergence density [7]. In the limit of slow motion,  $\rho \gg p/c^2$ , and  $p/c^2$  can be ignored so that  $\rho_t = \rho/2$ , and Eq.(22) gives

$$\nabla^2\phi = 4\pi G\rho \quad (24)$$

The reason why this discrepancy has not been recognized is that in finding the Schwarzschild metric we usually solve  $R_{\mu\nu} = 0$  as field equation. We may conclude that the form of the Einstein field equations with cosmological constant consistent with CP is

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) - \Lambda g_{\mu\nu} \quad (25)$$

This field equation may be derived from standard actions by considering the density metric of weight +1 instead of the metric as dynamical variables which is defined as [8]:

$$\tilde{g}_{\mu\nu} = \sqrt{-g} g_{\mu\nu} \quad (26)$$

and we get

$$\delta I = \int d^4x \left\{ \frac{c^4}{16\pi G} (R^{\mu\nu} + \Lambda g^{\mu\nu}) + \frac{1}{2} (T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T) \right\} \delta \tilde{g}_{\mu\nu} \quad (27)$$

From (26) we have

$$\delta \tilde{g}_{\mu\nu} = \sqrt{-g} \delta g_{\mu\nu} - \frac{1}{2} \sqrt{-g} g_{\mu\nu} g^{\alpha\beta} \delta g_{\alpha\beta} \quad (28)$$

Inserting (28) in (27) gives the ordinary variation of standard action with respect to the variation of the metric.

$$\delta I = \int d^4x \left\{ \frac{c^4}{16\pi G} (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu}) + \frac{1}{2} T^{\mu\nu} \right\} \sqrt{-g} \delta g_{\mu\nu} \quad (29)$$

This procedure may be carried out in an elegant way by applying the Palatini approach based on the idea of treating the metric (the density metric) and the connection separately as dynamical variables which the variation with respect to the connection reveals that the connection is necessarily the metric connection.

It is evident from (29) that the common field equations (1) are obtained under the variations of  $\delta g_{\mu\nu}$  with the condition that  $|g| \neq 0$ . While the consistent form (20) are resulted from (27) under the variations of  $\delta \tilde{g}_{\mu\nu}$  without any condition.

## 5. REMARKS

Let us summarize the significant results.



- 1 - It is shown that how (c) statement may be explicitly obtained from (c) statement in the mentioned CP classification without violating PC.
- 2 - Einstein field equations in the common form (1) of Einstein tensor proportional to the energy-momentum tensor do not fulfill the CP from (c) point of view.
- 3 - The unimodular gravity field equations (13) do not satisfy the (c) statement.
- 4 - The alternative field equations (20) which are mathematically equivalent to the Einstein common field equations (1) satisfy the CP commitments completely. This means that indeed these two forms are not physically equivalent. In Cartesian spatial coordinates the Poisson equation may be obtained from all the components of this form of field equations.
- 5 - The failure of unimodular model in this study ceases the interpretation of the cosmological constant as an integration constant, i.e. it is a universal constant of nature.
- 6 - Derivation of Eq.(1) from Lagrangian formalism (29) requires the constraint  $|g| \neq 0$ . Thus the resulted field equations are restricted and are not necessarily defined for the whole space.
- 7 - By taking the density metric tensors (26) as dynamical variables the obtained field equations from Lagrangian formalism (27) are free from any constraint and holds everywhere.

Accordingly, we should accept to carry out recasting of the GR field equations.

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